Investigation of The Wigner Function and Quantum Entanglement

Gerard Duff
Supervised by Dr Michael Tuite (School of Mathematical Sciences)

Personal Details
I came to DIT Kevin St. in 1995 to study physics and mathematics. Having completed 3 years of the BSc (Applied Sciences) course, which included courses in management studies and German, I was sidelined for personal reasons for a further 3 years. I made a return to college in September 2001 for my final year. I have thoroughly enjoyed my time in the DIT and have found life in Kevin St. quite easy to adjust to, even the second time around. I am interested in all aspects of mathematics, applied mathematics and physics and in the teaching of those subjects.

Project Summary
My final year project, proposed by the School of Mathematics, is mathematical physics based. I was concerned primarily with a probability distribution function called the Wigner function, (Eugene Wigner 1932), a version of which has applications in a range of quantum technologies including quantum optics and quantum computing. The aims of the project were to (i) carry out a study of the Wigner function $W(p, q)$ and its properties, (ii) to investigate some of its applications to, for example, coherent and entangled states of light.

I began by studying the elements that make up the Wigner function itself. Those elements are, phase space, Fourier transforms, and the density matrix. Phase space means simply position and momentum space. Many physical situations lend themselves to analysis in phase space; e.g. Heisenberg’s Uncertainty Principle $\Delta p \cdot \Delta x = \hbar/4\pi$ is an expression in phase space, which is fundamental to physics. The density matrix is a very powerful tool in applied maths. Knowledge of the density matrix gives a full picture of the state of the entire system as a statistical or ‘ensemble’ average. The version of the density matrix used here also contains information about quantum expectation values. It also allows us to predict the value of an arbitrary observable $A$, by means of the commutator $[A, \rho]$.

$W(p, q)$ is a Fourier transform, and is given by Equation (1) below. Its inverse transform is given by Equation (2).

\[
W(p,q) = \int_{-\infty}^{\infty} \text{exp}(-ipr/\hbar)\langle x|\hat{p}|x'\rangle
\]

(1)

\[
\langle x|\hat{p}|x'\rangle = \overline{\psi(x)\psi(x')} = \frac{1}{\hbar} \int dp \text{exp}(ipr/\hbar)W(p,q)
\]

(2)

The Wigner function gives a full description of the state of the system in terms of $|\psi(x, t)|^2$ where $\psi(x, t)$ is the solution to the time dependent Schrödinger equation. In this way the Wigner function has been used to reconstruct $|\psi(x, t)|^2$ for

Figure 1. An example of the Wigner function where the $x$-$y$ plane corresponds to the phase plane $(p, q)$, and the $z$-axis corresponds to $W(p, q)$.
many different quantum states.

One such state is known as a coherent state. Coherent states are regarded as being quasiclassical. For example, a coherent state harmonic oscillator is analogous to the motion of a pendulum. The particle or wave of finite length is said to cohere, in the sense that it sticks together.

Figures 1 and 2 are graphs of two possible Wigner functions, where the x-y plane corresponds to the phase plane \((p, q)\), and the z-axis corresponds to \(W(p, q)\).

The Wigner function of a coherent state (Fig. 1) is a Gaussian bell curve and is positive everywhere. Such a state is called a minimum uncertainty state or M.U.S. Such states hold a special place in quantum mechanics, since they give an “equals to” in Heisenberg’s relations.

An entangled state (Fig. 2) however has negative fringes in its Wigner function. This is due to ‘interference terms’ which appear in \(W(p, q)\). For example, the Wigner function of Young’s double slit experiment is similar to Fig. 2. The interference terms have no classical analogue. It is for this reason that the magnitude of any negativities in the Wigner function are indicators of, and used to explore the so called classical-quantum boundary.

The Wigner function is called a quasiprobability distribution function because of the presence of these negative values.

The project deals with entanglement in a more mathematical way, via what is referred to as a tensor product, also the idea of a quantum bit or qubit is discussed.